Problem A.25

Consider the matrices

$$\mathsf{A} = \begin{pmatrix} 1 & 4 & 1 \\ 4 & -2 & 4 \\ 1 & 4 & 1 \end{pmatrix}, \qquad \mathsf{B} = \begin{pmatrix} 1 & -2 & -1 \\ -2 & 2 & -2 \\ -1 & -2 & 1 \end{pmatrix}.$$

(a) Verify that they are diagonalizable and that they commute.

(b) Find the eigenvalues and eigenvectors of A and verify that its spectrum is nondegenerate.

(c) Show that the eigenvectors of A are eigenvectors of B as well.

Solution

Notice that A and B have all real elements and are equal to their respective transposes. This means A and B are normal matrices and hence diagonalizable.

$$\begin{aligned} \left[\mathsf{A}^\dagger, \mathsf{A}\right] &= \mathsf{A}^\dagger\mathsf{A} - \mathsf{A}\mathsf{A}^\dagger \\ &= \widetilde{\mathsf{A}^*}\mathsf{A} - \mathsf{A}\widetilde{\mathsf{A}^*} \\ &= \widetilde{\mathsf{A}}\mathsf{A} - \mathsf{A}\widetilde{\mathsf{A}} \\ &= \mathsf{A}\mathsf{A} - \mathsf{A}\mathsf{A} \\ &= 0 \\ &\Rightarrow \quad \mathsf{A} \text{ is diagonalizable.} \end{aligned} \qquad \begin{aligned} \left[\mathsf{B}^\dagger, \mathsf{B}\right] &= \mathsf{B}^\dagger\mathsf{B} - \mathsf{B}\mathsf{B}^\dagger \\ &= \widetilde{\mathsf{B}^*}\mathsf{B} - \mathsf{B}\widetilde{\mathsf{B}^*} \\ &= \widetilde{\mathsf{B}}\mathsf{B} - \mathsf{B}\widetilde{\mathsf{B}} \\ &= \mathsf{B}\mathsf{B} - \mathsf{B}\mathsf{B} \\ &= 0 \\ &\Rightarrow \quad \mathsf{B} \text{ is diagonalizable.} \end{aligned}$$

Since

$$\mathsf{AB} = \begin{pmatrix} -8 & 4 & -8 \\ 4 & -20 & 4 \\ -8 & 4 & -8 \end{pmatrix} = \mathsf{BA},$$

the matrices, A and B, commute, meaning they can be simultaneously diagonalized.
Solve the eigenvalue problem for A.

$$\mathsf{Aa} = \lambda\mathsf{a}$$

Bring $\lambda\mathsf{a}$ to the left side and combine the terms.

$$(\mathsf{A} - \lambda\mathsf{I})\mathsf{a} = 0 \tag{1}$$

Since $\mathsf{a} \neq 0$, the matrix in parentheses must be singular, that is,

$$\det(\mathsf{A} - \lambda\mathsf{I}) = 0$$

$$\begin{vmatrix} 1-\lambda & 4 & 1 \\ 4 & -2-\lambda & 4 \\ 1 & 4 & 1-\lambda \end{vmatrix} = 0.$$

Write out the determinant and solve the equation for λ.

$$(1-\lambda)\begin{pmatrix}-2-\lambda & 4 \\ 4 & 1-\lambda\end{pmatrix} - 4\begin{pmatrix}4 & 4 \\ 1 & 1-\lambda\end{pmatrix} + 1\begin{pmatrix}4 & -2-\lambda \\ 1 & 4\end{pmatrix} = 0$$

$$(1-\lambda)[(-2-\lambda)(1-\lambda)-16] - 4[4(1-\lambda)-4] + 1[16-(-2-\lambda)(1)] = 0$$

$$36\lambda - \lambda^3 = 0$$

$$\lambda(6+\lambda)(6-\lambda) = 0$$

$$\lambda = \{-6, 0, 6\}$$

Because there are three distinct eigenvalues, $\lambda_- = -6$ and $\lambda_0 = 0$ and $\lambda_+ = 6$, for this 3×3 matrix, there will be one eigenvector corresponding to each. This means the collection of eigenvalues (the spectrum) is nondegenerate.

To find the corresponding eigenvectors, plug λ_-, λ_0, and λ_+ back into equation (1).

$$(\mathsf{A}-\lambda_-\mathsf{I})\mathsf{a}_- = 0 \qquad (\mathsf{A}-\lambda_0\mathsf{I})\mathsf{a}_0 = 0 \qquad (\mathsf{A}-\lambda_+\mathsf{I})\mathsf{a}_+ = 0$$

$$\begin{pmatrix}7 & 4 & 1\\ 4 & 4 & 4\\ 1 & 4 & 7\end{pmatrix}\begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \qquad \begin{pmatrix}1 & 4 & 1\\ 4 & -2 & 4\\ 1 & 4 & 1\end{pmatrix}\begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \qquad \begin{pmatrix}-5 & 4 & 1\\ 4 & -8 & 4\\ 1 & 4 & -5\end{pmatrix}\begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\left.\begin{aligned}7a_1+4a_2+a_3&=0\\ 4a_1+4a_2+4a_3&=0\\ a_1+4a_2+7a_3&=0\end{aligned}\right\} \qquad \left.\begin{aligned}a_1+4a_2+a_3&=0\\ 4a_1-2a_2+4a_3&=0\\ a_1+4a_2+a_3&=0\end{aligned}\right\} \qquad \left.\begin{aligned}-5a_1+4a_2+a_3&=0\\ 4a_1-8a_2+4a_3&=0\\ a_1+4a_2-5a_3&=0\end{aligned}\right\}$$

$$\left.\begin{aligned}a_3&=-7a_1-4a_2\\ a_1+a_2+a_3&=0\\ a_1+4a_2+7a_3&=0\end{aligned}\right\} \qquad \left.\begin{aligned}a_1&=-4a_2-a_3\\ 2a_1-a_2+2a_3&=0\\ a_1+4a_2+a_3&=0\end{aligned}\right\} \qquad \left.\begin{aligned}a_3&=5a_1-4a_2\\ a_1-2a_2+a_3&=0\\ a_1+4a_2-5a_3&=0\end{aligned}\right\}$$

$$a_1+a_2+(-7a_1-4a_2)=0 \qquad 2(-4a_2-a_3)-a_2+2a_3=0 \qquad a_1-2a_2+(5a_1-4a_2)=0$$

$$-6a_1-3a_2=0 \qquad -9a_2=0 \qquad 6a_1-6a_2=0$$

$$\begin{aligned}-2a_1&=a_2\\ a_3=-7a_1-4(-2a_1)&=a_1\end{aligned} \qquad \begin{aligned}a_2&=0\\ a_1=-4(0)-a_3&=-a_3\end{aligned} \qquad \begin{aligned}a_1&=a_2\\ a_3=5(a_2)-4a_2&=a_2\end{aligned}$$

$$\mathsf{a}_- = \begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix} = \begin{pmatrix}a_1\\ -2a_1\\ a_1\end{pmatrix} \qquad \mathsf{a}_0 = \begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix} = \begin{pmatrix}-a_3\\ 0\\ a_3\end{pmatrix} \qquad \mathsf{a}_+ = \begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix} = \begin{pmatrix}a_2\\ a_2\\ a_2\end{pmatrix}$$

Therefore, the eigenvectors corresponding to $\lambda_- = -6$ and $\lambda_0 = 0$ and $\lambda_+ = 6$ are respectively

$$\mathsf{a}_- = a_1\begin{pmatrix}1\\ -2\\ 1\end{pmatrix} \quad \text{and} \quad \mathsf{a}_0 = a_3\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix} \quad \text{and} \quad \mathsf{a}_+ = a_2\begin{pmatrix}1\\ 1\\ 1\end{pmatrix},$$

where a_1, a_2, and a_3 are arbitrary (due to the fact that the eigenvalue problem is homogeneous).

In order to diagonalize A, let S^{-1} be the 3×3 matrix whose columns are the eigenvectors with $a_1 = a_2 = a_3 = 1$ for simplicity.

$$\mathsf{S}^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Determine S, the similarity matrix, by finding the inverse of S^{-1}.

$$\left(\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ -2 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \\ -2 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \\ 0 & -2 & 3 & 2 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right)$$

$$\sim \left(\begin{array}{rrr|rrr} 1 & -1 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right)$$

$$\sim \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right)$$

Consequently,

$$\mathsf{S} = \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Compute SAS^{-1} and verify that A is diagonalizable.

$$\begin{aligned}\mathsf{SAS}^{-1} &= \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 \\ 4 & -2 & 4 \\ 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} -6 & 0 & 6 \\ 12 & 0 & 6 \\ -6 & 0 & 6 \end{pmatrix} \\ &= \begin{pmatrix} -6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix}\end{aligned}$$

Compute SBS^{-1} and verify that B is diagonalizable.

$$\begin{aligned}\mathsf{SBS}^{-1} &= \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & -2 & -1 \\ -2 & 2 & -2 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 4 & -2 & -2 \\ -8 & 0 & -2 \\ 4 & 2 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}\end{aligned}$$

Are these the eigenvalues of B along the main diagonal? Let's find out.

$$\mathsf{Bb} = \mu\mathsf{b}$$

Bring $\mu\mathsf{b}$ to the left side and combine the terms.

$$(\mathsf{B} - \mu\mathsf{I})\mathsf{b} = \mathsf{0} \tag{2}$$

Since $\mathsf{b} \neq \mathsf{0}$, the matrix in parentheses must be singular, that is,

$$\det(\mathsf{B} - \mu\mathsf{I}) = 0$$

$$\begin{vmatrix} 1-\mu & -2 & -1 \\ -2 & 2-\mu & -2 \\ -1 & -2 & 1-\mu \end{vmatrix} = 0.$$

Write out the determinant and solve the equation for μ.

$$(1-\mu)\begin{pmatrix} 2-\mu & -2 \\ -2 & 1-\mu \end{pmatrix} + 2\begin{pmatrix} -2 & -2 \\ -1 & 1-\mu \end{pmatrix} - 1\begin{pmatrix} -2 & 2-\mu \\ -1 & -2 \end{pmatrix} = 0$$

$$(1-\mu)[(2-\mu)(1-\mu)-4] + 2[-2(1-\mu)-2] - 1[4+(2-\mu)] = 0$$

$$-16 + 4\mu + 4\mu^2 - \mu^3 = 0$$

$$(4-\mu)(\mu+2)(\mu-2) = 0$$

$$\mu = \{-2, 2, 4\}$$

Verify that the eigenvectors of A are also those of B.

$$\mathsf{Ba}_- = \begin{pmatrix} 1 & -2 & -1 \\ -2 & 2 & -2 \\ -1 & -2 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ -2a_1 \\ a_1 \end{pmatrix} = \begin{pmatrix} 4a_1 \\ -8a_1 \\ 4a_1 \end{pmatrix} = 4\begin{pmatrix} a_1 \\ -2a_1 \\ a_1 \end{pmatrix} = 4\mathsf{a}_-$$

$$\mathsf{Ba}_0 = \begin{pmatrix} 1 & -2 & -1 \\ -2 & 2 & -2 \\ -1 & -2 & 1 \end{pmatrix}\begin{pmatrix} -a_3 \\ 0 \\ a_3 \end{pmatrix} = \begin{pmatrix} -2a_3 \\ 0 \\ 2a_3 \end{pmatrix} = 2\begin{pmatrix} -a_3 \\ 0 \\ a_3 \end{pmatrix} = 2\mathsf{a}_0$$

$$\mathsf{Ba}_+ = \begin{pmatrix} 1 & -2 & -1 \\ -2 & 2 & -2 \\ -1 & -2 & 1 \end{pmatrix}\begin{pmatrix} a_2 \\ a_2 \\ a_2 \end{pmatrix} = \begin{pmatrix} -2a_2 \\ -2a_2 \\ -2a_2 \end{pmatrix} = -2\begin{pmatrix} a_2 \\ a_2 \\ a_2 \end{pmatrix} = -2\mathsf{a}_+$$

The eigenvector corresponding to $\mu_- = -2$ is

$$\mathsf{b}_- = a_2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

the eigenvector corresponding to $\mu_0 = 2$ is

$$\mathsf{b}_0 = a_3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

and the eigenvector corresponding to $\mu_+ = 4$ is

$$\mathsf{b}_+ = a_1\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$$

where a_1, a_2, and a_3 are arbitrary.

Now prove it by plugging in $\mu_- = -2$, $\mu_0 = 2$, and $\mu_+ = 4$ into equation (2).

$$(\mathsf{B} - \mu_-\mathsf{I})\mathsf{b}_- = 0 \qquad (\mathsf{B} - \mu_0\mathsf{I})\mathsf{b}_0 = 0 \qquad (\mathsf{B} - \mu_+\mathsf{I})\mathsf{b}_+ = 0$$

$$\begin{pmatrix} 3 & -2 & -1 \\ -2 & 4 & -2 \\ -1 & -2 & 3 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -1 & -2 & -1 \\ -2 & 0 & -2 \\ -1 & -2 & -1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -3 & -2 & -1 \\ -2 & -2 & -2 \\ -1 & -2 & -3 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 3b_1 - 2b_2 - b_3 &= 0 \\ -2b_1 + 4b_2 - 2b_3 &= 0 \\ -b_1 - 2b_2 + 3b_3 &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} -b_1 - 2b_2 - b_3 &= 0 \\ -2b_1 + 0b_2 - 2b_3 &= 0 \\ -b_1 - 2b_2 - b_3 &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} -3b_1 - 2b_2 - b_3 &= 0 \\ -2b_1 - 2b_2 - 2b_3 &= 0 \\ -b_1 - 2b_2 - 3b_3 &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} b_3 &= 3b_1 - 2b_2 \\ -b_1 + 2b_2 - b_3 &= 0 \\ -b_1 - 2b_2 + 3b_3 &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} b_1 + 2b_2 + b_3 &= 0 \\ b_1 &= -b_3 \\ b_1 + 2b_2 + b_3 &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} 3b_1 + 2b_2 + b_3 &= 0 \\ b_2 &= -b_1 - b_3 \\ b_1 + 2b_2 + 3b_3 &= 0 \end{aligned}\right\}$$

$$-b_1 - 2b_2 + 3(3b_1 - 2b_2) = 0 \qquad (-b_3) + 2b_2 + b_3 = 0 \qquad b_1 + 2(-b_1 - b_3) + 3b_3 = 0$$

$$8b_1 - 8b_2 = 0 \qquad 2b_2 = 0 \qquad -b_1 + b_3 = 0$$

$$\begin{aligned} b_2 &= b_1 \\ b_3 &= 3b_1 - 2(b_1) = b_1 \end{aligned} \qquad \begin{aligned} b_2 &= 0 \\ b_1 &= -b_3 \end{aligned} \qquad \begin{aligned} b_3 &= b_1 \\ b_2 &= -b_1 - (b_1) = -2b_1 \end{aligned}$$

$$\mathsf{b}_- = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_1 \\ b_1 \end{pmatrix} \qquad \mathsf{b}_0 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} -b_3 \\ 0 \\ b_3 \end{pmatrix} \qquad \mathsf{b}_+ = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ -2b_1 \\ b_1 \end{pmatrix}$$

Therefore, the eigenvectors corresponding to $\mu_- = -2$ and $\mu_0 = 2$ and $\mu_+ = 4$ are respectively

$$\mathsf{b}_- = B_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathsf{b}_0 = B_2\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathsf{b}_+ = B_3\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$$

where B_1, B_2, and B_3 are arbitrary (due to the fact that the eigenvalue problem is homogeneous).